

# Degrees of Freedom of Two-Hop Wireless Networks: “Everyone Gets the Entire Cake”

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**Abstract**—We show that fully connected two-hop wireless networks with  $K$  sources,  $K$  relays and  $K$  destinations have  $K$  degrees of freedom for almost all values of constant channel coefficients. Our main contribution is a new interference-alignment-based achievability scheme which we call *aligned network diagonalization*. This scheme allows the data streams transmitted by the sources to undergo a diagonal linear transformation from the sources to the destinations, thus being received free of interference by their intended destination.

## I. INTRODUCTION

Recent years have seen a dramatic increase in the wireless data traffic, caused by the success of online media streaming services and the proliferation of smart phones, tablets, and netbooks. Given the scarcity of wireless spectrum, the only way to meet this ever increasing demand is to exploit a much denser spatial reuse of the spectrum through the utilization of user-deployed and user-operated infrastructure, such as residential femtocells. As a result, understanding how to communicate optimally in multi-hop multi-flow wireless networks becomes fundamental for the future of wireless communications.

Despite the increasing interest in technologies that enable multi-hop multi-flow wireless networks, our knowledge about their fundamental performance limits is still limited. A large body of work focuses on multi-user systems with point-to-point interfering links, i.e., interference channels. While the capacity of the interference channel remains unknown (except for special cases, such as [1–7]), it has been approximated to within a constant gap in the two-user case [8]. For the general  $K$ -user interference channel, it is known that, for almost all values of channel gains,  $K/2$  degrees of freedom are achievable both when the channel gains are fixed and when they are time-varying [9, 10]. These results established the foundations of a technique known as *interference alignment*, which provided a promising new way of handling interference in multi-user wireless networks. While a simple time-division scheme only grants each of the  $K$  users of an interference channel  $1/K$  degrees of freedom, interference alignment allows each user to achieve half a degree of freedom. Thus, even though all  $K$  users must share the wireless medium, each one can still achieve half of the degrees of freedom it would achieve if it had the channel entirely to itself; i.e., each of the  $K$  users can still get “half of the cake”.

In the realm of multi-hop wireless networks, most of the

results concern single-flow networks, where a single source node wishes to communicate with one or more destinations. For these networks, the multicast capacity was characterized to within a constant number of bits in [11], and tighter capacity approximations were later derived in [12, 13]. However, no general capacity results are known for multi-hop multi-flow wireless networks, and most of the recent work has considered specific network topologies. In [14], for instance, the authors considered a network with two sources, two relays and two destinations (the  $2 \times 2 \times 2$  interference channel, or  $2 \times 2 \times 2$  wireless network). By introducing a new scheme called *aligned interference neutralization*, which extends the idea of real interference alignment to a multi-hop scenario, they showed that these networks have two degrees of freedom for almost all values of the channel gains.

More general networks with two source-destination pairs were later considered in [15]. In this work, two new notions were introduced. The first one is the idea of network condensation, by which a network with an arbitrary number of layers is reduced to a network with at most four layers with the same degrees of freedom. The second is a graph theoretic characterization of when the interference in a network is *manageable*, i.e., when all the interference can be simultaneously neutralized. These concepts allowed the degrees of freedom region of two-unicast layered networks with an arbitrary number of layers, and arbitrary connectivity between adjacent layers to be completely characterized. Surprisingly, it was shown that, for almost all values of the channel gains, these networks can only have 1,  $3/2$  or 2 sum degrees of freedom.

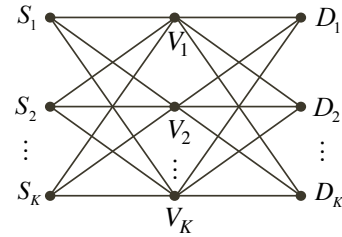


Fig. 1. The  $K \times K \times K$  Wireless Network.

When an arbitrary number of source-destination pairs  $K$  is considered, the results are scarcer. One such effort is found in [16], where networks with  $K$  source-destination pairs and  $K$  hops were considered under the fast fading scenario. The authors show that, under some assumptions on the joint

distribution of the channel gains,  $K$  degrees of freedom can be achieved. The main idea is to have the relays forward their received signals at carefully chosen times, so that the signals transmitted by the sources undergo an approximately diagonal end-to-end transformation. However, if the network has less than  $K$  hops, or if the channel coefficients are kept constant, the same ideas cannot be used.

In this paper, we focus on two-hop networks with  $K$  sources,  $K$  relays and  $K$  destinations – the  $K \times K \times K$  wireless network, shown in Figure 1. Several achievability schemes are known for this network, but, except in the case  $K = 2$ , they all fall short of the cut-set outer bound of  $K$  degrees of freedom. For example, if we consider linear schemes, we can use a result from [17] that shows that, in an  $N \times K \times N$  wireless network, interference can only be completely neutralized at all destinations if  $K \geq N(N-1)+1$ . Thus, it is possible to achieve  $\max\{N : K \geq N(N-1)+1\}$  (roughly  $\sqrt{K}$ ) degrees of freedom on the  $K \times K \times K$  wireless network, by using only a subset of  $N$  source-destination pairs. A different approach consists of viewing the

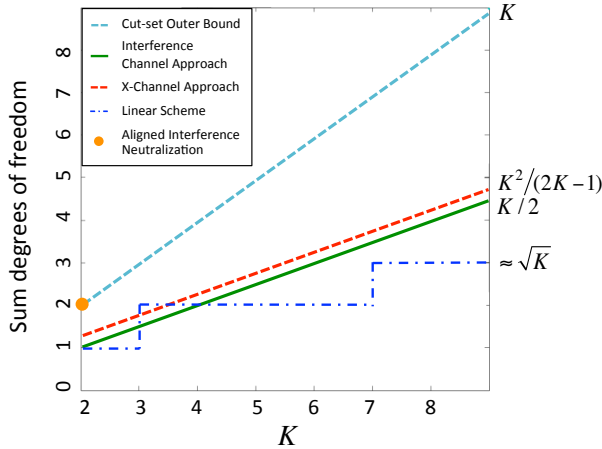


Fig. 2. Degrees of freedom achieved by different schemes on the  $K \times K \times K$  wireless network.

$K \times K \times K$  wireless network as the concatenation of two  $K$ -user interference channels. Since a  $K$ -user interference channel has  $K/2$  degrees of freedom for almost all values of the channel gains [10], we can also achieve  $K/2$  degrees of freedom on the  $K \times K \times K$  wireless network for almost all values of the channel gains. Similarly, we can view each hop of the  $K \times K \times K$  wireless network as a  $K$ -user X-channel. This approach in fact achieves  $K^2/(2K-1)$  degrees of freedom [18], which is slightly better than  $K/2$ . These three approaches are essentially the state-of-the-art in terms of inner bounds for the degrees of freedom of  $K \times K \times K$  wireless networks with  $K > 2$ , and, as depicted in Figure 2, their gap to the cut-set outer bound is essentially  $K/2$ , for large  $K$ . In [19], there has also been an attempt to find tighter upper bounds on the degrees of freedom of  $K \times K \times K$  networks, by using a recent worst-case noise theorem in

additive-noise wireless networks [20] to bound the capacity region of  $K \times K \times K$  Gaussian wireless networks with that of truncated deterministic networks. However, the best known upper bound on the sum degrees of freedom of  $K \times K \times K$  networks remains the cut-set upper bound (i.e.,  $K$  degrees of freedom).

In this work we close this gap by showing that the  $K \times K \times K$  wireless network has  $K$  degrees of freedom for almost all values of the channel gains. For this purpose, we introduce a new achievability scheme called *aligned network diagonalization*. Similar to the aligned interference neutralization scheme in [14], we let each source encode its message into several data streams, which are transmitted along distinct rational dimensions, using the real interference alignment framework [10]. However, the transmit directions at the sources and at the relays are carefully chosen so that the data streams transmitted by the sources essentially undergo a diagonal linear transformation until they reach the destinations. This way, interference-free channels are effectively created between each source and its corresponding destination, allowing each user to achieve arbitrarily close to one degree of freedom, i.e., each user can get “the entire cake”.

## II. PROBLEM SETUP

The  $K \times K \times K$  wireless network is made up of  $K$  sources  $S_1, \dots, S_K$ ,  $K$  relays  $V_1, \dots, V_K$ , and  $K$  destinations  $D_1, \dots, D_K$ , organized as a two-hop layered network, as shown in Figure 1. We assign channel gains  $h_{S_i, V_j}$  to the link  $(S_i, V_j)$ , for  $i, j \in \{1, \dots, K\}$ , and  $h_{V_i, D_j}$  to the link  $(V_i, D_j)$ . We assume that each channel gain is real-valued and fixed throughout the entire communication period. In Section V, we describe how our main results can be extended to the case where the channel gains vary with time.

Communication will take place over a block of  $n$  discrete time steps. At each time  $t = 1, 2, \dots, n$ , each node  $v \in \{S_1, \dots, S_K, V_1, \dots, V_K\}$  transmits a real-valued signal  $X_v[t]$ . The received signal at a relay  $V_j$  and at a destination  $D_j$  are respectively given by

$$Y_{V_j}[t] = \sum_{i=1}^K h_{S_i, V_j} X_{S_i}[t] + Z_{V_j}[t] \quad \text{and} \quad (1)$$

$$Y_{D_j}[t] = \sum_{i=1}^K h_{V_i, D_j} X_{V_i}[t] + Z_{D_j}[t], \quad (2)$$

where  $Z_{V_j}[t]$  and  $Z_{D_j}[t]$ , for  $t = 1, 2, \dots, n$ , are sequences of i.i.d. noise terms distributed as  $\mathcal{N}(0, \sigma^2)$ . The noise terms are also assumed to be independent from all transmit signals and noise terms at different nodes.

**Definition 1.** A coding scheme  $\mathcal{C}$  with block length  $n \in \mathbb{N}$  and rate tuple  $\mathbf{R} = (R_1, \dots, R_K) \in \mathbb{R}^K$  for the  $K \times K \times K$  wireless network consists of:

1. An encoding function  $f_i : \{1, \dots, 2^{nR_i}\} \rightarrow \mathbb{R}^n$  for each source  $S_i$ ,  $i = 1, \dots, K$ , where each codeword

$f_i(w_i)$ ,  $w_i \in \{1, \dots, 2^{nR_i}\}$ , satisfies an average power constraint of  $P$ .

2. Relaying functions  $r_i^{(t)} : \mathbb{R}^{t-1} \rightarrow \mathbb{R}$ , for  $t = 1, \dots, n$ , for each relay  $V_i$ ,  $i = 1, \dots, K$ , satisfying the average power constraint

$$\frac{1}{n} \sum_{t=1}^n \left[ r_i^{(t)}(y_1, \dots, y_{t-1}) \right]^2 \leq P,$$

for all  $(y_1, \dots, y_n) \in \mathbb{R}^n$ .

3. A decoding function  $g_i : \mathbb{R}^n \rightarrow \{1, \dots, 2^{nR_i}\}$  for each destination  $D_i$ ,  $i = 1, \dots, K$ .

**Definition 2.** The error probability of a coding scheme  $\mathcal{C}$  (as defined in Definition 1), is given by

$$P_{\text{error}}(\mathcal{C}) = \Pr \left[ \bigcup_{i=1}^K \{W_i \neq g_i(Y_{D_i}[1], \dots, Y_{D_i}[n])\} \right],$$

where we assume that each  $W_i$  is chosen independently and uniformly at random from  $\{1, \dots, 2^{nR_i}\}$ , that source  $S_i$  transmits  $f_i(W_i)$  over the  $n$  time-steps, and relay  $V_i$  transmits  $r_i^{(t)}(Y_{V_i}[1], \dots, Y_{V_i}[t-1])$  at time  $t = 1, \dots, n$ , for  $i = 1, \dots, K$ .

**Definition 3.** A rate tuple  $\mathbf{R}$  is said to be achievable for the  $K \times K \times K$  wireless network if there exists a sequence of coding schemes  $\mathcal{C}_n$  with rate tuple  $\mathbf{R}$  and blocklength  $n$ , for which  $P_{\text{error}}(\mathcal{C}_n) \rightarrow 0$ , as  $n \rightarrow \infty$ . The sequence of coding schemes  $\mathcal{C}_n$ ,  $n = 1, 2, \dots$ , is then said to achieve rate tuple  $\mathbf{R}$ .

**Definition 4.** The capacity region  $C(P)$  of a  $K \times K \times K$  wireless network is the closure of the set of achievable rate tuples, and the sum-capacity is defined as

$$C_{\Sigma}(P) = \max_{(R_1, \dots, R_K) \in C(P)} \sum_{i=1}^K R_i.$$

**Definition 5.** The degrees of freedom of a  $K \times K \times K$  wireless network are defined as

$$d_{\Sigma} = \lim_{P \rightarrow \infty} \frac{C_{\Sigma}(P)}{\frac{1}{2} \log P}.$$

### III. MAIN RESULTS

Our main result settles the question of the number of degrees of freedom of a  $K \times K \times K$  wireless network, for (Lebesgue) almost all values of the channel coefficients.

**Theorem 1.** For a  $K \times K \times K$  wireless network,  $d_{\Sigma} = K$  for almost all values of the channel gains.

Since the cut-set outer bound trivially implies that  $d_{\Sigma} \leq K$ , we only need to show that  $K$  degrees of freedom are achievable. Our achievability scheme utilizes the real interference alignment framework from [10] with carefully selected transmit directions. Each of the  $K$  sources will transmit a linear combination of  $L$  data streams. These

data streams are aligned at the relays, which allows each relay to decode approximately  $L$  linear combinations of the data streams which can then be re-modulated using new transmit directions. These new transmit directions are chosen so that all the interference is cancelled at each destination, and the  $L$  data streams from each source arrive at their intended destination along independent rational dimensions, which allows perfect decoding with high probability. These operations guarantee that, with small probability of error, the  $LK$  data streams chosen at all  $K$  sources are mapped to  $LK$  received directions at the destinations by a diagonal linear transformation. Hence, we call this scheme *aligned network diagonalization*.

The result in Theorem 1 has important consequences. Consider a two-hop  $K$ -unicast wireless network where, instead of having  $K$  relays, we have  $A$  relays; i.e., a  $K \times A \times K$  wireless network. It is easy to see that the cut-set outer bound states that no more than  $\min(K, A)$  degrees of freedom can be achieved. Now, if  $A \geq K$ , we can simply ignore  $A - K$  of the relays and use aligned network diagonalization to achieve  $K$  degrees of freedom. Similarly, if  $K > A$ , we can ignore  $K - A$  source-destination pairs, and achieve  $A$  degrees of freedom. A similar idea can be used in a multihop wireless network with  $J$  layers,  $K$  source-destination pairs and  $A_j$  relays in the  $j$ th layer (hence  $A_1 = A_J = K$ ). If we call such a network a  $K \times A_2 \times \dots \times A_{J-1} \times K$  wireless network, we have the following result.

**Corollary 1.** For a  $K \times A_2 \times \dots \times A_{J-1} \times K$  wireless network,  $d_{\Sigma} = \min_{1 \leq j \leq J} A_j$  for almost all values of the channel gains.

### IV. ACHIEVABILITY SCHEME

In this section we describe the scheme that achieves  $K$  degrees of freedom on a the  $K \times K \times K$  wireless network. We first describe the operations performed by the sources, relays and destinations, and then we do a performance analysis where we prove that the scheme in fact achieves  $K$  degrees of freedom.

#### A. Aligned Network Diagonalization Scheme Description

##### Encoding at the sources:

As in the real interference alignment schemes from [10] and the aligned interference neutralization scheme from [14], each source  $S_i$  starts by breaking its message  $W_i$  into  $L$  submessages. Each of the submessages will be encoded in a separate data stream, using a single codebook with codewords of length  $n$ , obtained by uniform i.i.d. sampling of the set

$$\mathcal{U} = \mathbb{Z} \cap \left[ -\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, \gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right], \quad (3)$$

for a small  $\epsilon > 0$ , a positive constant  $\gamma$ , and some positive integer  $M$  to be defined. The rate of this code, i.e., the number of codewords, will be determined later, in Section IV-B. Notice that  $M$  can be thought of as a parameter which

sets the number of degrees of freedom given to each stream to be  $(1-\epsilon)/(M+\epsilon) \approx 1/M$ . As explained in Section IV-B, we will choose  $M = (N+1)^{K^2}$ . Next, we let

$$T_{s_{11}, s_{12}, \dots, s_{KK}} = \prod_{\substack{1 \leq i \leq K \\ 1 \leq j \leq K}} h_{S_i, V_j}^{s_{ij}}, \quad (4)$$

and  $\Delta_N = \{0, \dots, N-1\}^{K^2}$ , and, by following the real interference alignment framework from [10], we define the set of *transmit directions* for the sources to be

$$\mathcal{T}_N = \{T_{s_{11}, s_{12}, \dots, s_{KK}} : (s_{11}, s_{12}, \dots, s_{KK}) \in \Delta_N\}, \quad (5)$$

for some arbitrary  $N$ . Notice that the number of transmit directions (which is also the number of data streams) is  $L = |\mathcal{T}_N| = |\Delta_N| = N^{K^2}$ . To simplify the notation we will let  $\vec{s}$  be a vector of indices  $(s_{11}, s_{12}, \dots, s_{KK})$  and write  $T_{\vec{s}}$ . We will let  $c_{i, \vec{s}}[1], c_{i, \vec{s}}[2], \dots, c_{i, \vec{s}}[n]$  be the  $n$  symbols of the codeword associated to the submessage to be sent by source  $S_i$  over the transmit direction indexed by  $\vec{s}$ . At time  $t \in \{1, \dots, n\}$ , source  $S_i$  will thus transmit

$$X_{S_i}[t] = \xi \sum_{\vec{s} \in \Delta_N} T_{\vec{s}} c_{i, \vec{s}}[t]$$

where  $\xi = \beta P^{\frac{M-1+2\epsilon}{2(M+\epsilon)}}$ , for a constant  $\beta$  to be determined. Since the maximum power of a transmit signal from  $S_i$  can be loosely upper bounded by

$$\beta^2 P^{\frac{M-1+2\epsilon}{M+\epsilon}} \left( \sum_{\vec{s} \in \Delta_N} |T_{\vec{s}}| \right)^2 \gamma^2 P^{\frac{1-\epsilon}{M+\epsilon}} = \beta^2 \left( \sum_{\vec{s} \in \Delta_N} |T_{\vec{s}}| \right)^2 \gamma^2 P,$$

for any value of  $\gamma$  and  $N$ , we can choose the constant  $\beta$  such that the maximum transmit power at the sources is no more than  $P$ .

### Relaying operations:

The received signal at relay  $V_j$  can be written as

$$Y_{V_j}[t] = \xi \sum_{\vec{s} \in \Delta_N} T_{\vec{s}} \left( \sum_{i=1}^K h_{S_i, V_j} c_{i, \vec{s}}[t] \right) + Z_{V_j}[t]. \quad (6)$$

Even though writing the received signal as in (6) does not emphasize the alignment that occurs at the relays, it will still be a useful representation of the received signal. To capture the alignment, we consider rearranging the terms in the summation in (6) by viewing it as a polynomial on the variables  $h_{S_i, V_j}$ , for  $1 \leq i, j \leq K$ , where the (integer) coefficients are given by sums of  $c_{i, \vec{s}}$  terms. It can then be seen that the actual set of received directions at each relay is a subset of  $\mathcal{T}_{N+1}$ , and the received signal at relay  $V_j$  at time  $t$  can be alternatively written as

$$Y_{V_j}[t] = \xi \sum_{\vec{s} \in \Delta_{N+1}} T_{\vec{s}} u_{j, \vec{s}}[t] + Z_{V_j}[t], \quad (7)$$

where  $u_{j, (s_{11}, s_{12}, \dots, s_{KK})}[t] = \sum_{i=1}^K c_{i, \vec{s}}[t]$  and we define  $c_{i, \vec{s}}[t] = 0$  if any component of  $\vec{s}$  is  $-1$  or  $N$ . Therefore, the (noiseless) received constellation at each relay is given by

$$\mathcal{V} = \left\{ \xi \sum_{\vec{s} \in \Delta_{N+1}} T_{\vec{s}} u_{\vec{s}} : u_{\vec{s}} \in \mathbb{Z} \cap \left[ -K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right], \forall \vec{s} \in \Delta_{N+1} \right\}. \quad (8)$$

Each relay  $V_j$  will map its received signal  $Y_{V_j}[t]$  to the nearest point in  $\mathcal{V}$ . This point can then be used to obtain the integers  $u_{j, \vec{s}}$ , for  $\vec{s} \in \Delta_{N+1}$ , due to the following claim (which is proven in Section IV-B).

**Claim 1.** *There exists a one-to-one map between points  $v \in \mathcal{V}$  and tuples of integers  $(u_{\vec{s}} : \vec{s} \in \Delta_{N+1})$  with entries in  $\mathbb{Z} \cap \left[ -K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right]$  such that  $v = \xi \sum_{\vec{s} \in \Delta_{N+1}} T_{\vec{s}} u_{\vec{s}}$ .*

After decoding  $u_{j, \vec{s}}$ , for  $\vec{s} \in \Delta_{N+1}$ , using this one-to-one map, relay  $V_j$  will re-encode all these integers using new transmit directions. To describe the new set of transmit directions, we first define

$$\begin{bmatrix} b_{11} & \cdots & b_{K1} \\ b_{12} & \cdots & b_{K2} \\ \vdots & \ddots & \vdots \\ b_{1K} & \cdots & b_{KK} \end{bmatrix} = \begin{bmatrix} h_{V_1, D_1} & \cdots & h_{V_K, D_1} \\ h_{V_1, D_2} & \cdots & h_{V_K, D_2} \\ \vdots & \ddots & \vdots \\ h_{V_1, D_K} & \cdots & h_{V_K, D_K} \end{bmatrix}^{-1}. \quad (9)$$

Notice that, for almost all values of the channel gains, the matrix on the right-hand side is invertible, and the  $b_{ij}$ s are well defined. Next, we let

$$\tilde{T}_{s_{11}, s_{12}, \dots, s_{KK}} = \prod_{\substack{1 \leq i \leq K \\ 1 \leq j \leq K}} b_{ij}^{s_{ij}}, \quad (10)$$

and, similar to (5), we can define the set of transmit directions for the relays to be

$$\tilde{\mathcal{T}}_{N+1} = \left\{ \tilde{T}_{s_{11}, s_{12}, \dots, s_{KK}} : (s_{11}, s_{12}, \dots, s_{KK}) \in \Delta_{N+1} \right\}.$$

Relay  $V_j$  will re-encode the  $u_{j, \vec{s}}$ s by essentially replacing each received direction  $T_{\vec{s}}$  in (7) with the direction  $\tilde{T}_{\vec{s}}$ . More precisely, the transmit signal of relay  $V_j$  at time  $t+1$  will be given by

$$X_{V_j}[t+1] = \xi' \sum_{\vec{s} \in \Delta_{N+1}} \tilde{T}_{\vec{s}} u_{j, \vec{s}}[t], \quad (11)$$

where  $\xi' = \beta' P^{\frac{M-1+2\epsilon}{2(M+\epsilon)}}$ , and  $\beta'$  is chosen so that the output power constraint is satisfied (similar to  $\beta$ ). We then have the following claim.

**Claim 2.** *The transmit signal of relay  $V_j$ , given in (11), can be re-written as*

$$X_{V_j}[t+1] = \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} \left( \sum_{i=1}^K b_{ij} c_{i, \vec{s}}[t] \right). \quad (12)$$

*Proof:* The main idea is to notice that, just as (7) can be written as (6), (11) can be re-written as (12). This can be more easily understood if we think of the (noiseless) received signal in (7) as a polynomial on the variables  $h_{S_i, V_j}$ ,  $1 \leq i, j \leq K$ , with integer coefficients. When relay  $V_j$  decodes each coefficient  $\tilde{u}_{j, \vec{s}}$  of this polynomial and then replaces each monomial  $T_{\vec{s}}$  with  $\tilde{T}_{\vec{s}}$ , it is essentially re-building the same polynomial with each variable  $h_{S_i, V_j}$  replaced by  $b_{ij}$ . Therefore, the same factorization used on the polynomial on the  $h_{S_i, V_j}$  variables in (6) can be used on the polynomial on the  $b_{ij}$  variables, as shown in (12). ■

### Decoding at the destinations:

In order to compute the received signals at the destinations, we first notice that, from (12), the vector of the  $K$  relay transmit signals at time  $t + 1$  can be written as

$$\begin{bmatrix} X_{V_1}[t+1] \\ \vdots \\ X_{V_K}[t+1] \end{bmatrix} = \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} \begin{bmatrix} b_{11} & \cdots & b_{K1} \\ \vdots & \ddots & \vdots \\ b_{1K} & \cdots & b_{KK} \end{bmatrix} \begin{bmatrix} c_{1, \vec{s}}[t] \\ \vdots \\ c_{K, \vec{s}}[t] \end{bmatrix}. \quad (13)$$

Since the  $\tilde{T}_{\vec{s}}$ s are just scalars, we can write the vector of the  $K$  received signals at the destinations as

$$\begin{aligned} & \begin{bmatrix} Y_{D_1}[t+1] \\ \vdots \\ Y_{D_K}[t+1] \end{bmatrix} \\ &= \begin{bmatrix} h_{V_1, D_1} & \cdots & h_{V_K, D_1} \\ \vdots & \ddots & \vdots \\ h_{V_1, D_K} & \cdots & h_{V_K, D_K} \end{bmatrix} \begin{bmatrix} X_{V_1}[t+1] \\ \vdots \\ X_{V_K}[t+1] \end{bmatrix} + \begin{bmatrix} Z_{D_1}[t+1] \\ \vdots \\ Z_{D_K}[t+1] \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & \cdots & b_{K1} \\ \vdots & \ddots & \vdots \\ b_{1K} & \cdots & b_{KK} \end{bmatrix}^{-1} \begin{bmatrix} X_{V_1}[t+1] \\ \vdots \\ X_{V_K}[t+1] \end{bmatrix} + \begin{bmatrix} Z_{D_1}[t+1] \\ \vdots \\ Z_{D_K}[t+1] \end{bmatrix} \\ &= \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} \begin{bmatrix} c_{1, \vec{s}}[t] \\ \vdots \\ c_{K, \vec{s}}[t] \end{bmatrix} + \begin{bmatrix} Z_{D_1}[t+1] \\ \vdots \\ Z_{D_K}[t+1] \end{bmatrix}. \end{aligned}$$

Thus, the received signal at destination  $D_j$  at time  $t + 1$  is simply given by

$$Y_{D_j}[t+1] = \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} c_{j, \vec{s}}[t] + Z_{D_j}[t+1], \quad (14)$$

and we see that all the interference has been cancelled, and destination  $D_j$  receives only the data streams originated at source  $S_j$ . The points in the (noiseless) received constellation at each destination, given by

$$\tilde{\mathcal{V}} = \left\{ \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} c_{\vec{s}} : c_{\vec{s}} \in \mathcal{U}, \forall \vec{s} \in \Delta_N \right\}, \quad (15)$$

can also be uniquely mapped into tuples of integers due to the following claim.

**Claim 3.** *There exists a one-to-one map between points  $v \in \tilde{\mathcal{V}}$  and tuples of integers  $(c_{\vec{s}} : \vec{s} \in \Delta_N)$  with entries in  $\mathcal{U}$  such that  $v = \xi' \sum_{\vec{s} \in \Delta_N} \tilde{T}_{\vec{s}} c_{\vec{s}}$ .*

Therefore, at each time  $t = 2, \dots, n$ , destination  $D_i$  will first map its received signal to the nearest point in  $\tilde{\mathcal{V}}$  and then use the one-to-one map between points in  $\tilde{\mathcal{V}}$  and tuples  $(c_{\vec{s}} : \vec{s} \in \Delta_N)$  with entries in  $\mathcal{U}$  to obtain the  $L$  integers  $c_{i, \vec{s}}$  encoded by source  $S_i$  at time  $t - 1$ . At time  $n$ , destination  $D_i$  has decoded  $L$  data streams of  $n$  integers each (in fact,  $n - 1$  integers, since the integers encoded by the destination at time  $t = n$  do not arrive at the destination within the length- $n$  block), and it applies an individual typicality-based decoder to each of these streams to decode the original source message  $W_i$ .

The step-by-step transformation that is induced by the aligned network diagonalization scheme is illustrated in Figure 3. For simplicity, we do not show the time indices nor the scaling coefficients  $\xi$  (whose purpose is to make sure the transmit power constraints are satisfied). A high-level overview of the scheme is as follows. All sources start by picking the same set of transmit directions, chosen in accordance to the first hop transfer matrix  $H_{S, V}$ . The choice of transmit directions guarantees that the ratio between the number of transmit directions at each source and the number of received directions at each relay

$$\frac{|\Delta_N|}{|\Delta_{N+1}|} = \left( \frac{N}{N+1} \right)^{K^2}$$

can be made arbitrarily close to one, giving us nearly perfect alignment over the first hop. The signal received by each relay  $V_j$  is a polynomial on the variables  $h_{S_i, V_j}$ , for  $1 \leq i, j \leq K$  with integer coefficients  $u_{j, \vec{s}}$ , for  $\vec{s} \in \Delta_{N+1}$ , plus a noise term. By decoding each integer coefficient, the relays can then rebuild these polynomials, where each variable  $h_{S_i, V_j}$  is replaced with the variable  $b_{ij}$  as defined in (9). This replacement of variables is equivalent to changing the directions along which each of the integer coefficients is being received. The key idea behind aligned network diagonalization is that, after the received directions at the relays are replaced, the relays are transmitting *what they would have received if the transfer matrix of the first hop had been  $H_{V, D}^{-1}$  instead of  $H_{S, V}$* . Therefore, we are effectively converting the transfer matrix of the first hop to  $H_{V, D}^{-1}$ , which causes the network to be diagonalized.

### B. Performance Analysis

In this section we show that the aligned network diagonalization scheme can in fact achieve  $K$  degrees of freedom. In order to do that, we first need to bound the error probability of the hard-decoding operations at the relays and destinations. In the process of doing that, we prove Claims 1 and 3.

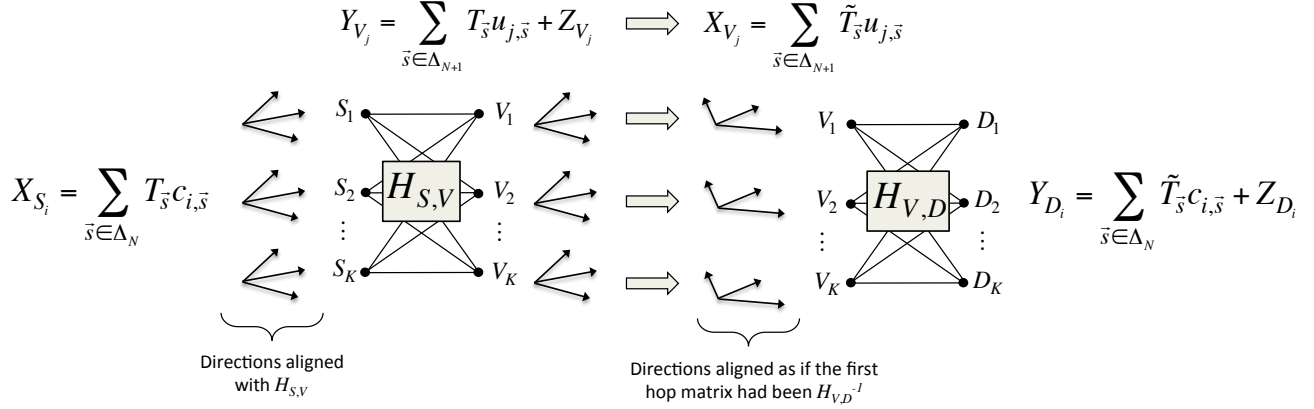


Fig. 3. Illustration of aligned network diagonalization.

### Relaying operations:

To bound the error probability of the relaying operations, we need to find a lower bound on the minimum distance between two points in the received constellation  $\mathcal{V}$ , described in (8). Since the directions  $T_{\vec{s}}$ , for  $\vec{s} \in \Delta_{N+1}$ , are all distinct monomials of the channel gains of the first hop, they can be viewed as analytic functions of  $h_{S_i, V_j}$ , for  $1 \leq i, j \leq K$ , that are linearly independent over the reals. Moreover, the distance between any two points in  $\mathcal{V}$  has the form

$$\xi \sum_{\vec{s} \in \Delta_{N+1}} T_{\vec{s}} u_{\vec{s}},$$

where each  $u_{\vec{s}}$  can take values in  $\mathbb{Z} \cap \left[ -2K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, 2K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right]$ . Thus, we can apply Theorem 5 in [10] (see also its subsequent remarks and inequality (8) in particular) to conclude that, for almost all values of the channel gains, there exists a constant  $\kappa$ , independent of  $P$ , such that the minimum distance of  $\mathcal{V}$  satisfies

$$d_{\min} > \xi \frac{\kappa}{\left( 2K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right)^{|\Delta_{N+1}| - 1 + \epsilon}}.$$

By choosing  $M = |\Delta_{N+1}| = (N+1)K^2$ , we have

$$d_{\min} > \frac{\kappa \beta P^{\frac{M-1+2\epsilon}{2(M+\epsilon)}}}{(2K\gamma)^{M-1+\epsilon} P^{\frac{(1-\epsilon)(M-1+\epsilon)}{2(M+\epsilon)}}} = \frac{\kappa \beta}{(2K\gamma)^{M-1+\epsilon}} P^{\epsilon/2}. \quad (16)$$

The fact that the minimum distance between any two points in  $\mathcal{V}$  is strictly positive implies that there exists a one-to-one map between points  $v \in \mathcal{V}$  and tuples of integers  $(u_{\vec{s}} : \vec{s} \in \Delta_{N+1})$  with entries in  $\mathbb{Z} \cap \left[ -K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, K\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right]$ , thus proving Claim 1. Therefore, after mapping its received signal to the nearest point in  $\mathcal{V}$ , relay  $V_j$  can in fact decode each  $u_{j,\vec{s}}$ ,  $\vec{s} \in \Delta_{N+1}$ , using this one-to-one map. This procedure will correctly decode each  $u_{j,\vec{s}}$ , provided that

$|Z_{V_j}[t]| < d_{\min}/2$ , implying that the probability of error for relay  $V_j$  is at most

$$\begin{aligned} \Pr(|Z_{V_j}[t]| \geq d_{\min}/2) &= 2Q\left(\frac{d_{\min}}{2\sigma}\right) \\ &\leq \exp\left(-\frac{d_{\min}^2}{8\sigma^2}\right) = \exp(-\delta P^\epsilon), \end{aligned} \quad (17)$$

where  $\delta$  is a positive constant that is independent of  $P$ .

### Decoding at the destinations:

Similar to what we did for the received signals at the relays, we would like to lower bound the minimum distance between two points in the destinations (noiseless) received constellation  $\tilde{\mathcal{V}}$ , given in (15). The following lemma, whose proof we present in the appendix, allows us to use Theorem 5 from [10] as we did before.

**Lemma 1.** *The received directions at the destinations,  $\tilde{T}_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$ , are analytic functions of  $h_{V_i, D_j}$ ,  $1 \leq i, j \leq K$ , that are linearly independent over the reals.*

Theorem 5 from [10] now implies that the minimum distance  $\tilde{d}_{\min}$  between any two points in  $\tilde{\mathcal{V}}$  can be lower-bounded as

$$\tilde{d}_{\min} > \xi' \frac{\tilde{\kappa}}{\left( 2\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}} \right)^{|\Delta_N| - 1 + \epsilon}}.$$

for some constant  $\tilde{\kappa}$  (which is independent of  $P$ ). Since  $M = |\Delta_{N+1}| > |\Delta_N|$ , for  $P > 1$ , we have

$$\tilde{d}_{\min} > \frac{\tilde{\kappa} \beta' P^{\frac{M-1+2\epsilon}{2(M+\epsilon)}}}{(2\gamma)^{|\Delta_N| - 1 + \epsilon} P^{\frac{(1-\epsilon)(M-1+\epsilon)}{2(M+\epsilon)}}} = \frac{\tilde{\kappa} \beta'}{(2\gamma)^{|\Delta_N| - 1 + \epsilon}} P^{\epsilon/2}. \quad (18)$$

The fact that the minimum distance between any two points in  $\tilde{\mathcal{V}}$  is strictly positive implies that there exists a one-to-one map between points  $v \in \tilde{\mathcal{V}}$  and tuples of integers

$(c_{\vec{s}} : \vec{s} \in \Delta_N)$  with entries in  $\mathbb{Z} \cap \left[-\gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}, \gamma P^{\frac{1-\epsilon}{2(M+\epsilon)}}\right]$ , thus proving Claim 3. After mapping its received signal to the nearest point in  $\tilde{\mathcal{V}}$ , destination  $D_j$  can in fact decode each  $c_{j,\vec{s}}, \vec{s} \in \Delta_N$ , using this one-to-one map. As in (17), the probability that  $D_i$  incorrectly decodes these integers (provided that no relay made an error in the previous step) is at most

$$\Pr(|Z_{D_j}[t]| \geq \tilde{d}_{\min}/2) = \exp(-\tilde{\delta}P^\epsilon), \quad (19)$$

for some constant  $\tilde{\delta} > 0$ .

### Achievable rates:

To determine the rate of our original codebook, we first notice that each data stream between  $S_i$  and  $D_i$  effectively creates a discrete memoryless channel with input and output alphabets  $\mathcal{U}$  and an error probability which can be upper bounded as

$$\begin{aligned} P_e &\leq 1 - (1 - \exp(-\delta P^\epsilon))^K (1 - \exp(-\tilde{\delta}P^\epsilon)) \\ &\leq 1 - (1 - \exp(-\delta' P^\epsilon))^{K+1} \\ &\leq (K+1) \exp(-\delta' P^\epsilon), \end{aligned} \quad (20)$$

where  $\delta' = \min(\delta, \tilde{\delta})$ . This allows us to lower bound the mutual information between the input  $U$  and the output  $\hat{U}$  of this channel, for a uniform distribution over the input alphabet. Using Fano's inequality, we have

$$\begin{aligned} I(U; \hat{U}) &\geq H(U) - H(U|\hat{U}) \\ &\geq \log |\mathcal{U}| - (1 + P_e \log |\mathcal{U}|) \\ &= (1 - P_e) \log |\mathcal{U}| - 1 \\ &\geq (1 - (K+1) \exp(-\delta' P^\epsilon)) \\ &\quad \left( \frac{1-\epsilon}{M+\epsilon} \frac{\log P}{2} + \log 2\gamma \right) - 1, \end{aligned}$$

and we can achieve rate

$$R = (1 - (K+1) \exp(-\delta' P^\epsilon)) \left( \frac{1-\epsilon}{M+\epsilon} \frac{\log P}{2} + \log 2\gamma \right) - 1$$

over each data stream, by having our original codebook have  $2^{nR}$  codewords. This means that each data stream can achieve

$$\lim_{P \rightarrow \infty} \frac{R}{\frac{1}{2} \log P} = \frac{1-\epsilon}{M+\epsilon} = \frac{1-\epsilon}{(N+1)^{K^2} + \epsilon}$$

degrees of freedom. Since each source transmits  $L = |\Delta_N| = N^{K^2}$  data streams, each source-destination pair achieves a total of

$$\frac{(1-\epsilon)N^{K^2}}{(N+1)^{K^2} + \epsilon} \geq \frac{(1-\epsilon)N^{K^2}}{(1+\epsilon)(N+1)^{K^2}} = \frac{1-\epsilon}{1+\epsilon} \left( \frac{N}{N+1} \right)^{K^2}$$

degrees of freedom, for any large  $N$  and any small  $\epsilon > 0$ , implying that each source-destination pair can achieve arbitrarily close to one degree of freedom. We conclude that the aligned network diagonalization scheme can achieve arbitrarily close to  $K$  degrees of freedom for almost all values of the channel gains, which proves Theorem 1.

## V. TIME-VARYING $K \times K \times K$ WIRELESS NETWORKS

In our problem setup in section II, we considered the case where the channel gains remain fixed throughout the entire communication period. Another important scenario is the case where the channel gains vary with time. In the case of time-varying channels, we let the channel gain between source  $S_i$  and relay  $V_j$  at time  $t$  be  $h_{S_i, V_j}[t]$ , and the channel gain between relay  $V_i$  and destination  $D_j$  at time  $t$  be  $h_{V_i, D_j}[t]$ , for  $t = 1, 2, \dots$ . We assume that  $\{h_{S_i, V_j}[t]\}_{t=1}^\infty$  and  $\{h_{V_i, D_j}[t]\}_{t=1}^\infty$  are all mutually independent i.i.d random processes each obeying an absolutely continuous probability distribution, and that instantaneous channel state information is available at all nodes.

One possible way of extending the aligned network diagonalization scheme to the time-varying scenario is to simply utilize the scheme described in Section IV-A based on the channel gain values at each time-step. This would be done by replacing the transmit directions in (4) by

$$T_{s_{11}, s_{12}, \dots, s_{KK}}[t] = \prod_{\substack{1 \leq i \leq K \\ 1 \leq j \leq K}} h_{S_i, V_j}[t]^{s_{ij}},$$

and the transmit direction in (10) by

$$\tilde{T}_{s_{11}, s_{12}, \dots, s_{KK}}[t] = \prod_{\substack{1 \leq i \leq K \\ 1 \leq j \leq K}} b_{ij}[t]^{s_{ij}},$$

where the  $b_{ij}[t]$ s vary with time according to the channel gains of the second hop as

$$\begin{bmatrix} b_{11}[t] & \cdots & b_{K1}[t] \\ \vdots & \ddots & \vdots \\ b_{1K}[t] & \cdots & b_{KK}[t] \end{bmatrix} = \begin{bmatrix} h_{V_1, D_1}[t] & \cdots & h_{V_K, D_1}[t] \\ \vdots & \ddots & \vdots \\ h_{V_1, D_K}[t] & \cdots & h_{V_K, D_K}[t] \end{bmatrix}^{-1}.$$

Therefore, at each time  $t = 2, \dots, n$ , each destination  $D_i$  attempts to decode the the  $L$  integers  $c_{i,\vec{s}}$  encoded by source  $S_i$  at time  $t-1$  in the same way as it would in the case of constant channels. Thus, at the end of the block of  $n$  time steps, destination  $D_i$  can apply the same typicality-based decoder to the sequence of decoded integers to decode  $W_i$ ,  $i = 1, \dots, K$ .

## VI. CONCLUSION

In this work, we showed that, for almost all values of channel gains,  $K \times K \times K$  wireless networks have  $K$  degrees of freedom. This result is surprising due to the fact that, in a  $K \times K \times K$  wireless network, each destination is subject to interference originated at  $K-1$  sources. Thus, the total number of interference signals that need to be neutralized for  $K$  degrees of freedom to be achieved is  $O(K^2)$ , while the number of variables under our control (i.e., the encoding rules at the sources and the relaying operations) is only  $O(K)$ .

The coding scheme we propose – aligned network diagonalization – shows that a careful choice of transmit directions at the relays can in fact neutralize all the interference at the destinations. The scheme can be understood as taking



the vector of received signals at the relays and modifying them so that it “looks like” the transfer matrix of the first hop was the inverse of the transfer matrix of the second hop. This way, we can effectively diagonalize the network, creating parallel interference-free channels from each source to its corresponding destination, thus allowing each source-destination pair to achieve one degree of freedom.

## VII. ACKNOWLEDGEMENTS

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## APPENDIX

**Lemma 1.** *The received directions at the destinations,  $\tilde{T}_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$ , are analytic functions of  $h_{V_i, D_j}$ ,  $1 \leq i, j \leq K$ , that are linearly independent over the reals.*

*Proof:* To prove that each  $\tilde{T}_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$ , is an analytic function of  $h_{V_i, D_j}$ ,  $1 \leq i, j \leq K$ , we notice that if we let

$$H = \begin{bmatrix} h_{V_1, D_1} & h_{V_2, D_1} & \cdots & h_{V_K, D_1} \\ h_{V_1, D_2} & h_{V_2, D_2} & \cdots & h_{V_K, D_2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{V_1, D_K} & h_{V_2, D_K} & \cdots & h_{V_K, D_K} \end{bmatrix},$$

then, for  $1 \leq i, j \leq K$  we can write  $b_{ij} = \frac{C_{ij}}{\det H}$ , where  $C_{ij}$  is the cofactor of the  $(i, j)$  entry of  $H$ . This means that each  $b_{ij}$  is a ratio of two polynomials with  $h_{V_i, D_j}$ ,  $1 \leq i, j \leq K$ , as variables. Since each  $\tilde{T}_{\vec{s}}$  is a distinct monomial of the  $b_{ij}$ s, it is clear that each  $\tilde{T}_{\vec{s}}$  is an analytic function of  $h_{V_i, D_j}$ ,  $1 \leq i, j \leq K$ .

Next, suppose by contradiction that  $\tilde{T}_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$ , are not linearly independent over the reals. Then there must be real numbers  $\alpha_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$ , not all zero, such that

$$\sum_{\vec{s} \in \Delta_N} \alpha_{\vec{s}} \tilde{T}_{\vec{s}} = 0$$

for all values of  $h_{V_i, D_j}$ , for  $1 \leq i, j \leq K$ . However, since the  $\tilde{T}_{\vec{s}}$ , for  $\vec{s} \in \Delta_N$  are distinct monomials of the  $b_{ij}$ s, we have that, for almost all values of the  $b_{ij}$ s,  $\sum_{\vec{s} \in \Delta_N} \alpha_{\vec{s}} \tilde{T}_{\vec{s}} \neq 0$ . Since for almost all values of the  $b_{ij}$ s, the matrix

$$B = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{K1} \\ b_{12} & b_{22} & \cdots & b_{K2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1K} & b_{2K} & \cdots & b_{KK} \end{bmatrix}$$

is invertible, we can find  $b_{11}, b_{12}, \dots, b_{KK}$  for which  $B$  is invertible and  $\sum_{\vec{s} \in \Delta_N} \alpha_{\vec{s}} \tilde{T}_{\vec{s}} \neq 0$  (with the  $\tilde{T}_{\vec{s}}$ s seen as functions of the  $b_{ij}$ s). But this means that if we choose the values of  $h_{V_i, D_j}$ , for  $1 \leq i, j \leq K$ , by setting  $H = B^{-1}$ , we will have  $\sum_{\vec{s} \in \Delta_N} \alpha_{\vec{s}} \tilde{T}_{\vec{s}} \neq 0$  (with the  $\tilde{T}_{\vec{s}}$ s seen as functions of the  $h_{V_i, D_j}$ s), which is a contradiction. ■

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